Combining WDF Elements

With WDFs, it's pretty common to combine a series voltage source plus resistor into a "resistive voltage source", in order to be able to use a voltage source as an "adaptable" WDF one-port. However, we can also combine other WDF elements with the goal being that the C++ compiler can better optimize the computations needed for the combined elements.

Resistor-Capacitor Series

Here we create a combined one-port for a resistor and capacitor in series. Note that with the combined element we no longer have direct access to the voltage or current across either the resistor or capacitor, so this approach doesn't work great if you need to "probe" either element.

Underlying Components

A WDF Resistor is defined by:

$$R_p = R, b = 0$$

A WDF Capacitor is defined by:

$$R_p = \frac{T}{2C}, b = a[n-1]$$

And a 3-port WDF Series adaptor:

$$\begin{gathered} R_p = R_1 + R_2 \\ b_0 = -a_1 - a_2 \\ b_1 = -\frac{R_1}{R_p} a_0 + \frac{R_2}{R_p} a_1 - \frac{R_1}{R_p} a_2 \\ b_2 = -\frac{R_2}{R_p} a_0 - \frac{R_2}{R_p} a_1 + \frac{R_1}{R_p} a_2 \end{gathered}$$

We can simplify this into 1-multiply form. First, we recognize the similarity between the b_1 and b_2 terms:

$$\begin{gathered} b_1 + b_2 = -\frac{R_1}{R_p} a_0 - \frac{R_2}{R_p} a_0 \\ b_1 + b_2 = -\left(\frac{R_1 + R_2}{R_p} a_0 \right) \end{gathered}$$

Recall that $R_p = R_1 + R_2$, so:

$$\begin{gathered} b_1 + b_2 = -\left(\frac{R_1 + R_2}{R_1 + R_2} a_0 \right) \\ b_1 + b_2 = -a_0 \\ b_2 = -(a_0 + b_1) \end{gathered}$$

So if we can figure out b_1 then we can compute b_2 with zero multiplications.

$$b_1 = -\frac{R_1}{R_p}a_0 + \frac{R_2}{R_p}a_1 - \frac{R_1}{R_p}a_2$$
$$b_1 = -\frac{R_1}{R_p}a_0 + \frac{R_2}{R_p}a_1 - \frac{R_1}{R_p}a_2 + \frac{R_1}{R_p}a_1 - \frac{R_1}{R_p}a_1$$
$$b_1 = -\frac{R_1}{R_p}(a_0 + a_1 + a_2) + \frac{R_2}{R_p}a_1 + \frac{R_1}{R_p}a_1$$
$$b_1 = a_1 - \frac{R_1}{R_p}(a_0 + a_1 + a_2)$$

So the full 1-multiply series adaptor is:

$$R_p = R_1 + R_2$$
$$b_0 = -a_1 - a_2$$
$$b_1 = a_1 - \frac{R_1}{R_p}(a_0 + a_1 + a_2)$$
$$b_2 = -(a_0 + b_1)$$

Combining The Elements

Now we can combine these elements. We'll use port (1) of the series adaptor for the capacitor, and port (2) for the resistor. Remember that a and b are being described from the reference point of the given element, so a_1 for the series adaptor is actually b from the capacitor. To avoid duplicated names, we'll use b_c and a_c for the variables from the perspective of the capacitor and b_r and a_r for the resistor. In other words: $a_1 := b_c$, $b_1 := a_c$, and so on.

First we sub in b_c and b_r for a_1 and a_2:

$$R_p = \frac{T}{2C} + R$$
$$R_i := \frac{R_1}{R_p} = \frac{\frac{T}{2C}}{\frac{T}{2C} + R} = \frac{T}{T + 2RC}$$
$$b_0 = -b_c - b_r$$
$$b_1 = a_c[n-1] - R_i(a_0 + b_c + b_r)$$
$$b_2 = -(a_0 + b_1)$$

And then we can sub in the definitions $b_c = a_c[n-1]$ and $b_r = 0$:

$$b_0 = -a_c[n-1] - 0$$
$$b_1 = a_c[n-1] - R_i(a_0 + a_c[n-1] + 0)$$
$$b_2 = -(a_0 + b_1)$$

Now we can sub in a_c and a_r for b_1 and b_2:

$$b_0 = -a_c[n-1]$$
$$a_c = a_c[n-1] - R_i(a_0 + a_c[n-1])$$
$$a_r = -(a_0 + a_c)$$

Since the only thing we need to compute is b_0, and b_0 only depends on a_c, we no longer need to compute a_r, and we can rename a_c to z (since that's essentially the "state" for this one-port).

$$b_0 = -z[n-1]$$
$$z = z[n-1] - R_i(a_0 + z[n-1])$$

So what's the advantage here? The series adaptor needs roughly 7 additions/subtractions and 1 multiply. The combined element needs only 3 additions/subtractions (and still the 1 multiply), but we've also removed a layer of complexity that we were hoping the compiler would optimize through. This results in better generated assembly (see, e.g., https://godbolt.org/z/Esodbobh3), and probably faster compile times as well. For WDF trees that are very deep, simplifying these elements also makes it less likely that the compiler will "give up" part-way through the optimizing process because of time/memory constraints.

Resistor-Capacitor Parallel

Underlying Components

The 3-port parallel adaptor is defined:

$$G_p = G_1 + G_2$$
$$b_0 = \frac{G_1}{G_p}a_1 + \frac{G_2}{G_p}a_2$$
$$b_1 = a_0 - \frac{G_2}{G_p}a_1 + \frac{G_2}{G_p}a_2$$
$$b_2 = a_0 + \frac{G_2}{G_p}a_1 - \frac{G_2}{G_p}a_2$$

Again, we find a simple relationship between b_1 and b_2:

$$b_d := a_2 - a_1$$
$$b_1 = a_0 + \frac{G_2}{G_p}b_d$$
$$b_2 = a_0 - \frac{G_1}{G_p}b_d$$
$$b_1 - b_2 = \frac{G_2}{G_p}b_d + \frac{G_1}{G_p}b_d$$
$$b_1 - b_2 = \frac{G_1 + G_2}{G_p}b_d$$
$$b_1 - b_2 = \frac{G_1 + G_2}{G_1 + G_2}b_d$$
$$b_1 - b_2 = b_d$$

Then solving for b_0:

$$b_0 = \frac{G_1}{G_p}a_1 + \frac{G_2}{G_p}a_2 + \frac{G_1}{G_p}a_2 - \frac{G_1}{G_p}a_2$$
$$b_0 = \frac{G_1}{G_p}a_2 + \frac{G_2}{G_p}a_2 + \frac{G_1}{G_p}(a_1 - a_2)$$
$$b_0 = a_2 - \frac{G_1}{G_p}b_d$$

So the full 1-multiply parallel adaptor:

$$G_p = G_1 + G_2$$
$$b_d = a_2 - a_1$$
$$b_0 = a_2 - \frac{G_1}{G_p} b_d$$
$$b_1 = b_2 + b_d$$
$$b_2 = a_0 - \frac{G_1}{G_p} b_d = b_0 - a_2 + a_0$$

Combining The Elements

Again, we can combine the elements with the capacitor at port (1), and resistor at port (2):

$$G_p = \frac{2C}{T} + \frac{1}{R}$$
$$G_i := \frac{G_1}{G_p} = \frac{\frac{2C}{T}}{G_p} = \frac{2CR}{2CR + T}$$
$$b_d = b_r - b_c$$
$$b_0 = b_r - G_i b_d$$
$$b_1 = b_2 + b_d$$
$$b_2 = b_0 - b_r + a_0$$

b_d condenses very nicely, to $-a_c[n-1]$, so:

$$b_0 = 0 + G_i a_c[n-1]$$
$$b_1 = b_2 - a_c[n-1]$$
$$b_2 = b_0 - 0 + a_0$$

Then substituting in $b_1 \rightarrow a_c$ and $b_2 \rightarrow a_r$:

$$b_0 = G_i a_c[n-1]$$
$$a_c = a_r - a_c[n-1]$$
$$a_r = b_0 + a_0$$

Substituing in for a_r, we get:

$$a_c = b_0 + a_0 - a_c[n-1]$$

So the final derivation:

$$b_0 = G_i z[n-1]$$
$$z = b_0 + a_0 - z[n-1]$$

Resistive Voltage Source + Capacitor (in Series)

Underlying Components

A WDF Resistive Voltage source is defined:

$$R_p = R, b = V$$

Combining The Elements

Now we can start again with the 1-multiply series adaptor:

$$R_p = R_1 + R_2$$
$$b_0 = -a_1 - a_2$$
$$b_1 = a_1 - \frac{R_1}{R_p}(a_0 + a_1 + a_2)$$
$$b_2 = -(a_0 + b_1)$$

And start subbing in:

$$R_p = \frac{T}{2C} + R$$
$$R_i := \frac{R_1}{R_p} = \frac{T}{T + 2RC}$$
$$b_0 = -a_c[n-1] - V$$
$$a_c = a_c[n-1] - R_i(a_0 + a_c[n-1] + V)$$

Note that as with the previous series combination, we can basically ignore the a_r term. We can simplify a_c a little bit more by subbing in the definition of b_0:

$$a_c = a_c[n-1] - R_i(a_0 - b_0)$$

In final form:

$$b_0 = -z[n-1] - V$$
$$z = z[n-1] - R_i(a_0 - b_0)$$

Capacitive Voltage Source (Series)

Let's start with the Kirchoff domain definition for a Resistive Voltage Source:

$$v(t) - e(t) = i(t)R$$

Now we replace the resistor term with a capacitor, and switch to the s-domain:

$$V(s) - E(s) = \frac{I(s)}{Cs}$$

And then moving to the wave domain:

$$\frac{1}{2}(A(s) + B(s)) - E(s) = \frac{1}{2R_0}\frac{A(s) - B(s)}{Cs}$$

$$R_0Cs(A(s) + B(s)) - 2R_0CsE(s) = A(s) - B(s)$$

$$B(s)(R_0Cs + 1) = A(s)(1 - R_0Cs) + 2R_0CsE(s)$$

$$B(s) = A(s)\frac{1 - R_0Cs}{1 + R_0Cs} + E(s)\frac{2R_0Cs}{1 + R_0Cs}$$

Now we apply the bilinear transform: $s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$:

$$B(z) = A(z)\frac{1 - R_0 C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}{1 + R_0 C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} + E(z)\frac{2 R_0 C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}{1 + R_0 C \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

Multiplying through by $\frac{T(1+z^{-1})}{T(1+z^{-1})}$:

$$B(z) = A(z)\frac{T(1+z^{-1}) - 2R_0 C(1-z^{-1})}{T(1+z^{-1}) + 2R_0 C(1-z^{-1})} + E(z)\frac{4R_0 C(1-z^{-1})}{T(1+z^{-1}) + 2R_0 C(1-z^{-1})}$$

$$B(z) = A(z)\frac{(T - 2R_0 C) + (T + 2R_0 C)z^{-1}}{(T + 2R_0 C) + (T - 2R_0 C)z^{-1}} + E(z)\frac{4R_0 C - 4R_0 C z^{-1}}{(T + 2R_0 C) + (T - 2R_0 C)z^{-1}}$$

Now we can adapt the port by setting $R_0 = \frac{T}{2C}$ (note that this is the same port impedance as a singular capacitor). We can then simplify $2R_0 C \to T$

$$B(z) = A(z)\frac{2Tz^{-1}}{2T} + E(z)\frac{2 - 2z^{-1}}{2T}$$

$$B(z) = A(z)z^{-1} + E(z)(1 - z^{-1})$$

Applying the inverse z-transform:

$$b[n] = a[n-1] + e[n] - e[n-1]$$